Explanation of Twin Paradox & Equivalence Principle

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***Abstract*—The Twin's Paradox is generally explained by the asymmetry due to acceleration / frame shift, first we discuss this approach briefly. After which we will be analysing the situation in the perspective of the moving twin. At the end with the help of equivalence principle we compare (pseudo)-gravitational time dilation and time dilation due to uniformly accelerated frames.**

I. Traditional Approach

If we consider the normal twin paradox situation with $v = 0.8c$, distance to the travelling point $L_0 = 4$ light years and infinite acceleration at the turn around point. The space-time diagram with lines of simultaneity looks as in figure 1, with the time AB begin skipped over in the perspective of the moving twin (the one with proper acceleration) (now referred to as to as X') due to the rotation of the plane of simultaneity.

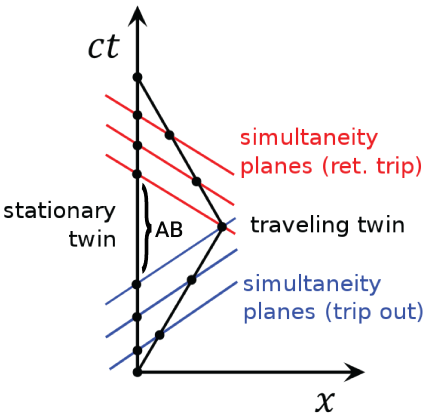

Fig. 1: Minkowski diagram of the twin paradox.

According to the stationary twin (with no proper acceleration) (now referred to as to as X) this trip will take

$$t_X^X = \frac{2L_0}{v} = 10\, years \tag{1}$$

where the superscript refers to the person whose reference frame we are measuring time in and the subscript refers to the person whose time/age we are measuring. Now using the Lorentz-transformation equations X would predict X' time as

$$t_{X'}^X = t_X^X \sqrt{1 - \frac{v^2}{c^2}} = 6\, years \tag{2}$$

II. In the reference frame of the moving twin X'

X' observes twin X and the starting point and the ending point move with a velocity $v = 0.8c$, the distance between the starting point and ending point will be Lorentz contracted as follows

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} = 2.4\, light\, years \tag{3}$$

According to X' time take by X to complete the trip i.e his aging during the trip would be

$$t_{X'}^{X'} = \frac{2L}{v} = 6\, years \tag{4}$$

This agrees with what X predicted. But due to the velocity-dependent time dilation X' would measure X age as

$$t_X^{X'} = t_{X'}^{X'} \sqrt{1 - \frac{v^2}{c^2}} = 6\sqrt{1 - \frac{v^2}{c^2}} = 3.6\, years \tag{5}$$

This is in disagreement with what X predicted. This is what the twin paradox is. It's solution is that we can't apply Lorentz transformation in the case where we consider X' to be a stationary observer because his metric cannot be Minkowski i.e he is in an non-inertial reference frame.

The metric for the uniformly accelerated frame is given by the Rindler metric

$$ds^2 = -\left(1 + \frac{ax}{c^2}\right)^2 c^2 dt^2 + dx^2 \tag{6}$$

Where a is the proper acceleration
Using the general physical representation of metric in a time-like space that it represents proper time τ for infinitesimal coordinate difference

$$ds^2 = -\left(1 + \frac{ax}{c^2}\right)^2 c^2 dt^2 + dx^2 = -c^2 d\tau^2 \tag{7}$$

Using $dx = v(x)dt$

$$-\left(1 + \frac{ax}{c^2}\right)^2 c^2 dt^2 + v(x)^2 dt^2 = -c^2 d\tau^2$$

$$d\tau = \int \sqrt{(1 + \frac{ax(t)}{c^2})^2 - \frac{v^2(t)}{c^2}} dt$$

The velocity at the turn-around point should be zero because the particle would need to reach a velocity zero for turning. This can also be seen via modeling the acceleration as a Dirac-delta function. Also this turn around would happen at a fixed location, here lets say the distance between X and X' is h, then

$$\Delta\tau = \left(1 + \frac{ah}{c^2}\right)\Delta t \tag{8}$$

Now to find Δt the time X' experiences a gravitational field, this is where this derivation is not rigorous and a better derivation using Lagrangian mechanics [1] can be used to get the same result we obtain using this,

$$a = \Delta v / \Delta t$$

$$\Delta t = 2v/a$$

Using this in eq.(8)

$$\Delta \tau = \left(1 + \frac{ah}{c^2}\right)\frac{2v}{a} = \frac{2v}{a} + \frac{2hv}{c^2}$$

In the limit $a \to \infty$

$$\Delta \tau = \frac{2hv}{c^2} = \frac{(2).(4c).(0.8c)}{c^2} = 6.4\, years \tag{9}$$

Hence now the total ageing is eq. (5) + eq. (9) which is $6.4 + 3.6 = 10$ years which matches up with what X predicted for himself

III. Comparison to the pseudo homogeneous gravitational field

Here in the analysis from the moving twin, to assert that he is at rest during the acceleration part, we can say that a pseudo homogeneous gravitational field has covered the entirety of space. (Application of the Equivalence Principle)

We know that there would be gravitational time dilation due to this, [2]

$$\Delta \tau = \left(1 + \frac{\Phi}{c^2}\right)\Delta t$$

where $\Phi = gh$ where g is the gravitational force and h is the distance between the objects.

$$\Delta \tau = \left(1 + \frac{gh}{c^2}\right)\Delta t \tag{10}$$

Thus this formula is identical to our derived formula in equation 8, with the help of equivalence principle.

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